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A new method for deriving rigorous results on $\pi\pi$ scattering

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Abstract. We develop a new approach to the problem of constraining the $\pi\pi$ scattering amplitudes by means of the axiomatically proved properties of unitarity, analyticity and crossing symmetry. The method is based on the solution of an extremal problem on a convex set of analytic functions and provides a global description of the domain of values taken by any finite number of partial waves at an arbitrary set of unphysical energies, compatible with unitarity, the bounds at complex energies derived from generalised dispersion relations and the crossing integral relations. From this domain we obtain new absolute bounds for the amplitudes as well as rigorous correlations between the values of various partial waves.

1. Introduction

During the last few years considerable progress has been made in the derivation of absolute bounds on pion–pion scattering. The last major improvement was achieved by Lopez and Mennessier (1977) who obtained a very impressive lower bound for the $\pi^0\pi^0$ S-wave scattering length, $a^{00} > -1{\cdot}75 m_\pi^{-1}$. In spite of this progress, the present bounds still remain largely outside the range of phenomenologically reasonable values and have a weak constraining power in practical situations. The problem of deciding whether these bounds can be further improved or are an ultimate limit of the axiomatic theory is therefore of much interest. Concerning the upper bounds it was shown recently by Auberson *et al* (1978), by means of an explicit construction, that the best bounds known up to now (Bonnier *et al* 1975, Lopez and Mennessier 1977) can be saturated. On the other hand, for the lower bounds the problem is still open and a further improvement of the present values is not excluded.

The present work is an attempt to investigate this kind of problem along a completely different line. We develop a formalism which provides a global description of the domain of values taken by a finite number of partial waves at a given set of coincident or different energies. The rigorous properties which we take into account in this description are analyticity, unitarity, the bounds at complex energies derived from generalised dispersion relations (Bonnier and Vinh Mau 1968, Bonnier 1975) and the crossing integral relations (Roskies 1970). Other rigorous constraints like positivity, convexity and monotony of the various partial waves, Martin inequalities (Martin 1967, Diţă 1973), as well as the numerical bounds known so far at particular points, can be simply introduced in this description. The method can be applied to every isospin amplitude as well as to any combination of them.

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The main point of our method is the formulation of the rigorous properties of the partial wave amplitudes in terms of an extremal problem in some space of analytic functions. This problem is formulated and solved in § 2. The applications of this problem to $\pi\pi$ scattering are discussed in § 3. For simplicity we restrict ourselves to $\pi^0\pi^0$ scattering and show first how one can cast the axiomatic properties of the partial waves in the canonical form in which the above-mentioned problem was formulated. Some numerical results concerning the S and D partial waves, as well as some further applications of the method, are presented at the end of the paper.

The work contains two appendices: in the first one, which is technical, we present the proof of a statement made in § 2. In appendix 2, starting from the bounds available for the total amplitude $F(s, t)$ at some particular points (Lopez and Mennessier 1977) we derive by a simple method new upper and lower bounds for the partial waves, which are useful for our analysis.

2. Solution of an extremal problem

In the present section we shall formulate and solve an extremal problem on a convex set of analytic functions. The relevance of this problem for the derivation of new rigorous results on $\pi\pi$ total and partial amplitudes will be discussed in the next section.

We shall consider functions $f(z)$ real analytic in the unit disc $|z|<1$ and belonging to the H^∞ Banach space (Duren 1970). Our problem will be to describe the domain

$$\mathscr{D}=\{f_i|f_i=f(z_i),\ i=1,\ldots,n, f\in\bar{S}_\infty\}$$

where $\bar{S}_\infty$ is the intersection of the unit sphere of H^∞ with a finite number of hyperplanes:

$$\bar{S}_\infty=\left\{f(z)|f(z)\in H^\infty, \|f\|_\infty\leqslant 1, \int_{-1}^{1}\psi_K(x)f(x)\mathrm{d}x=\alpha_K, K=1,\ldots,p\right\}.$$

For simplicity we shall consider here only real non-coincident points z_i. Then one can easily see that $\mathscr{D}$ is a closed and convex domain in $\mathbb{R}^n$. The generalisation to complex z_i, as well as the inclusion of derivatives of $f(z)$, is trivial. As will be shown explicitly in the next section the constraints defining the convex set $\bar{S}_\infty$ are the canonical transcription of the unitarity and crossing conditions for the $\pi\pi$ scattering amplitudes.

A solution in a closed form of the above problem was not actually obtained. Instead we were able to express the domain $\mathscr{D}$ as the intersection of a certain collection of closed and convex domains $\mathscr{D}_g$, each of them being determined exactly; namely let $\mathscr{D}_g$ be the domain

$$\mathscr{D}_g=\{f_i|f_i=f(z_i),\ i=1,\ldots,n, f\in\bar{S}_2(g)\}$$

where $\bar{S}_2(g)$ is a convex functional set described in the same way as the set $\bar{S}_\infty$ introduced above, with the only difference being that the H^∞-norm condition

$$\|f\|_\infty\leqslant 1 \tag{2.1}$$

entering the definition of $\bar{S}_\infty$ is replaced by the H^2 norm

$$\|gf\|_2\leqslant 1. \tag{2.2}$$

Here $g(z)$ is an outer function in the unit sphere of H^2 defined as

$$g(z) = \exp\left(\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln \rho(\theta)\right) \tag{2.3}$$

in terms of a positive weight function $\rho(\theta)$ obeying the conditions

$$\rho(\theta) \geqslant 0 \qquad \ln \rho(\theta) \in L^1(-\pi, \pi) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(\theta)\, d\theta \leqslant 1. \tag{2.4}$$

From the definition given above one can easily verify the relation $\mathscr{D} \subset \mathscr{D}_g$ for every $g(z)$ with the properties (2.3) and (2.4). Actually it can be shown that the stronger result $\mathscr{D} = \bigcap_g \mathscr{D}_g$ holds. The proof of this equality is given in appendix 1.

Our problem will therefore be the description of the domain $\mathscr{D}_g$ for a given $g(z)$. We mention that a similar embedding of an extremal problem for vector-valued functions from H^∞ into H^2 was considered also in connection with the derivation of sum rules for Compton scattering (Raszillier 1978, Auberson and Mennessier 1979).

The starting point of our derivation is the remark that the domain $\mathscr{D}_g$ is actually described analytically by the inequality

$$\mu_2(g, \{f_i\}, \{\alpha_K\}) \equiv \inf \|gf\|_2 \leqslant 1, \tag{2.5}$$

the infimum being taken on the subset of H^∞ defined by the constraints

$$\begin{aligned} &f(z_i) = f_i = \bar{f}_i \qquad i = 1, \ldots, n \\ &\int_{-1}^{1} f(x)\psi_K(x)\, dx = \alpha_K \qquad K = 1, \ldots, p. \end{aligned} \tag{2.6}$$

With this description of $\mathscr{D}_g$ its properties of convexity and closedness become transparent in connection with the convexity and continuity properties of the H^2 norm (2.5). We also point out that the frontier of $\mathscr{D}_g$ is given by the equality sign in (2.5). Accordingly, the problem is to find the expression of the minimal norm (2.5) with the constraints (2.6).

We notice that the first set of conditions (2.6) can be naturally taken into account by writing $f(z)$ as

$$f(z) = \sum_{K=1}^{n} A_K B_K(z) + B_{n+1}(z) h(z) \tag{2.7}$$

where

$$B_1(z) = 1 \qquad B_K(z) = B_{K-1}(z)\frac{z - z_{K-1}}{1 - z z_{K-1}} \qquad K = 2, \ldots, n+1$$

and A_i are determined in a non-singular way by solving the triangular system

$$f_i = \sum_{K=1}^{i} A_K B_K(z_i) \qquad i = 1, \ldots, n. \tag{2.8}$$

As for the function $h(z)$ appearing in (2.7), the second set of conditions (2.6) make it belong to the set $\mathscr{H}$ defined as

$$\mathscr{H}=\left\{h(z)\in H^{\infty}\middle|\int_{-1}^{1} B_{n+1}(x)\psi_K(x)h(x)\,\mathrm{d}x=\beta_K,\right. \\ \left.\beta_K=\alpha_K-\sum_{i=1}^{n} A_i\int_{-1}^{1}\psi_K(x)B_i(x)\,\mathrm{d}x,\ K=1,\ldots,p\right\}. \tag{2.9}$$

With the above notations the extremum problem to be solved becomes

$$\mu_2(g,\{f_i\},\{\alpha_K\})=\inf_{h\in\mathscr{H}}\|hg-kg\|_2 \tag{2.10}$$

where

$$k(\zeta)=-\sum_{K=1}^{n} A_K B_K(\zeta)/B_{n+1}(\zeta) \qquad \zeta=\mathrm{e}^{\mathrm{i}\theta} \tag{2.11}$$

is a rational function bounded for $\theta\in[-\pi,\pi]$. We shall solve (2.10) by applying the duality theorem for convex sets (Luenberger 1968, Duren 1970). We have

$$\inf_{h\in\mathscr{H}}\|hg-kg\|_2 \\ =\sup_{\substack{F\in L^2\\ \|F\|_2\leqslant 1}}\left|\frac{1}{2\pi\mathrm{i}}\oint F(\zeta)g(\zeta)k(\zeta)\,\mathrm{d}\zeta-\sup_{h\in\mathscr{H}}\frac{1}{2\pi\mathrm{i}}\oint F(\zeta)h(\zeta)g(\zeta)\,\mathrm{d}\zeta\right|. \tag{2.12}$$

In writing this equality we took into account Beurling's approximation theorem (Duren 1970), ensuring that the products $g(z)h(z)$ generate a dense set in H^2, for $h\in H^{\infty}$ and $g(z)$ defined by the relations (2.3) and (2.4). We shall first calculate the last term in the right-hand side of (2.12), which is actually the support functional of the convex set $\mathscr{H}$ (Luenberger 1968). By writing the Fourier series

$$F(\zeta)=\sum_{j=-\infty}^{\infty} F_j\zeta^j \qquad F\in L^2 \\ l(\zeta)\equiv g(\zeta)h(\zeta)=\sum_{j=0}^{\infty} l_j\zeta^j \qquad l\in H^2 \tag{2.13}$$

the support functional of $\mathscr{H}$ becomes

$$\sup_{h\in\mathscr{H}}\frac{1}{2\pi\mathrm{i}}\oint F(\zeta)g(\zeta)h(\zeta)\,\mathrm{d}\zeta=\sup_{h\in\mathscr{H}}\sum_{j=0}^{\infty} l_jF_{-(j+1)}. \tag{2.14}$$

Using the same notation the conditions defining the convex $\mathscr{H}$ may be written as

$$\sum_{j=0}^{\infty} l_jC_K^j=\beta_K \qquad K=1,\ldots,p \tag{2.15}$$

where

$$C_K^j=\int_{-1}^{1}\mathrm{d}x B_{n+1}(x)\psi_K(x)x^j/g(x).$$

If these p conditions are linearly independent then

$$\operatorname{rank}\|C_K^j\|_{K=1,\ldots,p}^{j=1,2,\ldots}=p$$

and we can assume that the determinant

$$\Delta = \det\begin{vmatrix} C_1^0 \cdots C_1^{p-1} \\ \vdots \quad\quad \vdots \\ C_p^0 \cdots C_p^{p-1} \end{vmatrix} \tag{2.16}$$

is different from zero.

From (2.12) one can see that only the finite values of the support functional are of interest for our problem. By comparing (2.14) and (2.15) it follows that this happens only when the relation (2.14) can be expressed as a linear combination of the expressions (2.15). This means that the equality

$$\operatorname{rank}\begin{Vmatrix} F_{-1} & F_{-2} & \cdots \\ & C_K^j & \end{Vmatrix} = p$$

must hold, which, taking into account the assumption following (2.16), gives the Fourier coefficients F_{-K} of $F(\zeta)$ as

$$F_{-(K+1)} = \frac{1}{\Delta}\sum_{j=1}^{p} (-1)^{j+1} F_{-j} D_j^K \qquad K = p, p+1, \ldots \tag{2.17}$$

where

$$D_j^k = \det\begin{vmatrix} C_1^K & C_1^0 & \ldots & C_1^{j-1} & C_1^{j+1} & \ldots & C_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ C_p^K & C_p^0 & \ldots & C_p^{j-1} & C_p^{j+1} & \ldots & C_p^{p-1} \end{vmatrix}. \tag{2.18}$$

By introducing (2.17) into (2.14) we obtain finally the support functional of $\mathscr{K}$ in the form

$$\sup_{h\in\mathscr{K}} \frac{1}{2\pi \mathrm{i}} \oint F(\zeta) g(\zeta) h(\zeta)\, \mathrm{d}\zeta = -\frac{1}{\Delta}\det\begin{vmatrix} 0 & F_{-1} & \ldots & F_{-p} \\ \beta_1 & C_1^0 & \ldots & C_1^{p-1} \\ \vdots & \vdots & & \vdots \\ \beta_p & C_p^0 & \ldots & C_p^{p-1} \end{vmatrix}. \tag{2.19}$$

We shall now evaluate the first term in the right-hand side of (2.12). To this end we shall take explicitly into account the fact that the conditions (2.17) restrict the form of $F(\zeta)$, more precisely of its non-analytic part, so that we can write

$$F(\zeta) = F_+(\zeta) + \sum_{j=1}^{p} F_{-j} Q_j \tag{2.20}$$

where $F_+(\zeta)$ is the analytic part of $F(\zeta)$:

$$F_+(\zeta) = \sum_{j=0}^{\infty} F_j \zeta^j \tag{2.21}$$

and

$$Q_j(\zeta) = 1/\zeta^j + \frac{(-1)^{j+1}}{\zeta^p \Delta}\det\begin{vmatrix} \int_{-1}^{1} \frac{\psi_1(x) x^p B_{n+1}(x)\,\mathrm{d}x}{g(x)(\zeta - x)} & C_1^0 & \ldots & C_1^{p-1} \\ \vdots & \vdots & & \vdots \\ \int_{-1}^{1} \frac{\psi_p(x) x^p B_{n+1}(x)\,\mathrm{d}x}{g(x)(\zeta - x)} & C_p^0 & \ldots & C_p^{p-1} \end{vmatrix}. \tag{2.22}$$

By applying the residue theorem, the analytic part $F_+(\zeta)$ gives in (2.12) the contribution

$$\sum_{i=1}^{n} A_i \sum_{j=i}^{n} F_+(z_j)g(z_j)[(\zeta-z_j)B_i(\zeta)/B_{n+1}(\zeta)]_{\zeta=z_j}$$

which, by changing the order of summation and using the relation (2.8) between A_i and f_j, may be written as

$$\sum_{j=1}^{n} F_+(z_j)f_jg(z_j)[(z-z_j)/B_{n+1}(z)]_{z=z_j}. \tag{2.23}$$

As concerns the non-analytic part of $F(\zeta)$, it gives in (2.12) the contribution

$$\sum_{j=1}^{p} F_{-j}R_j \tag{2.24}$$

where the numbers R_j are defined according to (2.20) as

$$R_j=\frac{1}{2\pi \mathrm{i}}\oint g(\zeta)k(\zeta)Q_j(\zeta)\,\mathrm{d}\zeta \qquad j=1,\ldots,p. \tag{2.25}$$

By adding the terms appearing in (2.19), (2.23) and (2.24) we can express the supremum (2.12) in the final form

$$\mu_2(g,\{f_i\},\{\alpha_K\})=\sup_{\substack{\|F\|_2\leq 1\\ F(\zeta)=F_+(\zeta)+\sum_{j=1}^{p}F_{-j}Q_j(\zeta)}}\left|\sum_{j=-p}^{\infty}F_j\xi_j\right| \tag{2.26}$$

where the coefficients ξ_j are

$$\xi_j=\sum_{K=1}^{n} z_j^K f_jg(z_j)[(z-z_j)/B_{n+1}(z)]_{z=z_j} \qquad j\geq 0$$

$$\xi_{-j}=R_j+\frac{(-1)^j}{\Delta}\begin{vmatrix}\beta_1 & C_1^0 & \ldots & C_1^{j-1} & C_1^{j+1} & \ldots & C_1^{p-1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \beta_p & C_p^0 & \ldots & C_p^{j-1} & C_p^{j+1} & \ldots & C_p^{p-1}\end{vmatrix} \qquad j=1,\ldots,p. \tag{2.27}$$

In order to evaluate the supremum (2.26) let us write explicitly the L^2 norm of $F(\zeta)$. By using (2.17) we have

$$\begin{aligned}\|F\|_2^2&=\sum_{K=-\infty}^{\infty}F_K^2=\sum_{K=0}^{\infty}F_K^2+\sum_{K=1}^{p}F_{-K}^2+\frac{1}{\Delta^2}\sum_{K=p}^{\infty}\sum_{i,j=1}^{p}(-1)^{i+j}F_{-i}F_{-j}D_i^KD_j^K\\ &=\sum_{K=0}^{\infty}F_K^2+\sum_{i,j=1}^{p}X_{ij}F_{-i}F_{-j}\leq 1.\end{aligned} \tag{2.28}$$

In the last term we introduced the matrix X defined as

$$X_{ij}=\delta_{ij}+\frac{(-1)^{i+j}}{\Delta^2}\sum_{l,m=1}^{p}M_{li}M_{mj}I_{lm} \qquad i,j=1,\ldots,p$$

M_{li} being the minor of the element situated on the lth row and ith column in the definition (2.16) of Δ and

$$I_{lm}=\int_{-1}^{1}\mathrm{d}x\int_{-1}^{1}\mathrm{d}y\frac{B_{n+1}(x)B_{n+1}(y)\psi_l(x)\psi_m(x)}{g(x)g(y)(1-xy)}x^py^p.$$

Let us denote by $\lambda_1, \ldots, \lambda_p$ the (non-negative) eigenvalues of the positive definite matrix X and by U the orthogonal matrix

$$U_{ij} = v_j^i \qquad i, j = 1, \ldots, p \tag{2.29}$$

where v^i are the eigenvectors of X:

$$Xv^i = \lambda_i v^i \qquad i = 1, \ldots, p.$$

If we now perform a finite change of coordinates by means of the matrix U, i.e. if we define

$$\begin{aligned} G_{-K} &= \sum_{j=1}^{p} (U^{-1})_{Kj} F_{-j} \qquad K = 1, \ldots, p \\ G_K &= F_K \qquad K \geqslant 0 \end{aligned} \tag{2.30}$$

we obtain the norm condition (2.28) in the form

$$\|F\|_2^2 = \sum_{K=0}^{\infty} G_K^2 + \sum_{K=1}^{p} \lambda_K G_{-K}^2 \leqslant 1 \tag{2.31}$$

while the supremum (2.26) is written as

$$\mu_2(g, \{f_i\}, \{\alpha_j\}) = \sup_{\|F\|_2 \leqslant 1} \left| \sum_{K=-p}^{\infty} G_K \eta_K \right| \tag{2.32}$$

with

$$\begin{aligned} \eta_K &= \xi_K \qquad K \geqslant 0 \\ \eta_K &= \sum_{l=1}^{p} U_{-Kl} \xi_{-l} \qquad K = -1, \ldots, -p. \end{aligned} \tag{2.33}$$

The evaluation of the supremum (2.32) with the condition (2.31) can be done immediately by applying the Cauchy–Schwarz inequality. We obtain

$$\mu_2(g, \{f_i\}, \{\alpha_j\}) = \left(\sum_{K=0}^{\infty} \eta_K^2 + \sum_{K=1}^{p} \eta_{-K}^2 / \lambda_K \right)^{1/2}. \tag{2.34}$$

Using the relations (2.33) and (2.27) the infinite sum in the last line can be written in a closed form:

$$\sum_{K=0}^{\infty} \eta_K^2 = \sum_{i,j=1}^{n} \frac{f_i f_j g(z_i) g(z_j)}{1 - z_i z_j} \left(\frac{z - z_i}{B_{n+1}(z)} \right)_{z=z_i} \left(\frac{z - z_j}{B_{n+1}(z)} \right)_{z=z_j} \tag{2.35}$$

which shows that the computation of $\mu_2(g, \{f_i\}, \{\alpha_j\})$ requires finally only a finite number of algebraic operations. The complete description of the closed and convex domain $\mathscr{D}_g$ is therefore given by the inequality

$$\sum_{i,j=1}^{n} \frac{f_i f_j g(z_i) g(z_j)}{1 - z_i z_j} \left(\frac{z - z_i}{B_{n+1}(z)} \right)_{z=z_i} \left(\frac{z - z_j}{B_{n+1}(z)} \right)_{z=z_j} + \sum_{K=1}^{p} \eta_{-K}^2 / \lambda_K \leqslant 1 \tag{2.36}$$

the expression on the left depending quadratically on the f_i, as may be seen going back again to equations (2.33) and (2.27).

As we have already mentioned, the domain $\mathscr{D}$ is obtained by intersecting all the above domains $\mathscr{D}_g$ for $g(z)$ subject to the conditions (2.3) and (2.4). In the present approach, performing this intersection amounts to taking the supremum upon $g(z)$ of the norms $\mu_2(g, \{f_i\}, \{\alpha_K\})$ which appear in the left-hand side of (2.36). This can be done explicitly for the case with no integral conditions (2.6), when (2.36) contains only the first term and the supremum upon $g(z)$ can be shown to provide exactly the known solution of the interpolation problem in H^∞ (Duren 1970, Krein and Nudelman 1973). In the present case the supremum of $\mu_2(g, \{f_i\}, \{\alpha_K\})$ cannot be obtained through a finite algorithm which, in other words, means that the initial H^∞ problem cannot be solved in a closed form. The above approach might have nevertheless a considerable practical utility. Indeed, each $\mathscr{D}_g$ is an approximant of $\mathscr{D}$ from outside and, with a fortunate choice of $g(z)$, one may hope to approach $\mathscr{D}$ rather closely. For instance, a useful guess proved to be

$$\begin{aligned} p(\theta) &= \frac{1-\alpha^2}{1-2\alpha\cos\theta+\alpha^2} \\ g(z) &= \frac{(1-\alpha^2)^{1/2}}{1-\alpha z} \end{aligned} \tag{2.37}$$

$\alpha \in (-1, 1)$ being a single parameter upon which the supremum in (2.36) has to be taken. This $p(\theta)$ is equal to the Jacobian of an arbitrary real mapping of the unit circle onto itself. The choice (2.37) is therefore equivalent to solving the problem in an infinity of H^2 norms, defined for all the unit discs obtained from each other by an arbitrary deformation of the frontier. By varying the parameter α one may expect to obtain a good approximation of $\mathscr{D}$. Numerical calculations for the interpolation problem revealed an excellent agreement between the H^2 and the H^∞ solution, which in this case is known, and offered useful suggestions as concerns the choice of the optimal α.

3. Application to the $\pi\pi$ scattering amplitudes

In this section we shall use the extremum problem solved above for obtaining new results about the $\pi\pi$ scattering amplitudes. Essentially, these results refer to the values taken by the partial waves at unphysical energies, compatible with analyticity, unitarity and crossing symmetry. Unlike the previous treatments of the same subject, our method is able to give simple and direct correlations between values taken at coincident or different points by various partial waves.

We shall start by recalling the rigorous properties of the $\pi\pi$ partial waves which enter the present approach. We shall then show how these properties can be written in the canonical form in which the extremum problem of § 2 was formulated.

For simplicity we shall treat here only the $\pi^0\pi^0$ scattering, the inclusion of the isospin being straightforward (Bonnier and Donohue 1978).

We shall consider the $\pi^0\pi^0$ partial waves $a_l(s)$ appearing in the development of the total $\pi^0\pi^0$ amplitude $F(s, t)$

$$F(s, t) = \sum_{l=0}^{\infty} (2l+1)a_l(s)P_l[1+2t/(s-4)] \tag{3.1}$$

and the S-matrix elements

$$S_l(s) = 1 - 2\left(\frac{4-s}{s}\right)^{1/2} a_l(s) \qquad l = 0, 2, \ldots. \tag{3.2}$$

Here s and t are the Mandelstam variables and the pion mass was taken equal to 1.

The axiomatic field theory predicts for the functions $S_l(s)$ a domain of analyticity in the complex s plane. For convenience we shall consider from this domain the disc

$$|s-26|\leqslant 26 \tag{3.3}$$

cut along the real axis between 4 and 52. In this disc $S_l(s)$ are real analytic functions and, along the physical cut $s\geqslant 4$, they satisfy the unitarity condition

$$|S_l(s)|\leqslant 1 \qquad 4\leqslant s\leqslant 52. \tag{3.4}$$

Furthermore, on the frontier of the disc (3.3), a rigorous upper bound for $|S_l(s)|$ is available, i.e.

$$|S_l(s)|\leqslant M_l(s) \qquad |s-26|=26. \tag{3.5}$$

This bound can be explicitly computed using (3.2) and the partial wave projection

$$a_l(s)=\int_0^1 F\left(s,\frac{\lambda(4-s)}{2}\right)P_l(1-\lambda)\,\mathrm{d}\lambda \tag{3.6}$$

taking into account the upper bound

$$|F(s,t)|\leqslant B(s,t) \tag{3.7}$$

derived for the total amplitude at complex s and t by Bonnier and Vinh Mau (1968) and Bonnier (1975), on the basis of dispersion relations on algebraic manifolds in the Mandelstam plane. The explicit form of $B(s,t)$ is found in the above references.

Actually, an upper bound of the type (3.5) can be computed directly for every complex s inside the analyticity domain. The choice of a large domain like (3.3) seems nevertheless to be advantageous, as it allows an additional use of the analyticity and unitarity condition (3.4). This is expected to improve our knowledge of the partial wave values, having in mind that the bounds (3.7) are far from exploiting the full content of the analyticity, unitarity and crossing symmetry.

Another rigorous property which we take into account in our approach is crossing symmetry. This is known to be equivalent to an infinite set of integral relations (Roskies 1970) involving a gradually increasing number of partial waves, in the form

$$\sum_{l=0}^{N}\int_0^4 \phi_{l,p}(s)q_l(s)\,\mathrm{d}s=0 \qquad p=0,1,\ldots. \tag{3.8}$$

For instance, the first two relations (3.8), containing only the S and D waves, are

$$\begin{aligned}&\int_0^4 (4-s)(3s-4)a_0(s)\,\mathrm{d}s=0\\ &\int_0^4 (4-s)^2[4(s-1)a_0(s)+(4-s)a_2(s)]\,\mathrm{d}s=0.\end{aligned} \tag{3.9}$$

For our purposes it is of interest to note that $\phi_{l,p}(s)$ are finite on $0\leqslant s\leqslant 4$ and vanish at $s=4$ at least like $(4-s)$.

The crossing relations (3.8) can be expressed, through (3.2), in terms of $S_l(s)$, giving a similar set of integral relations

$$\sum_{l=0}^{N}\int_0^4\left(\frac{s}{4-s}\right)^{1/2}\phi_{l,p}(s)S_l(s)\,\mathrm{d}s=\alpha_p \qquad p=0,1,\ldots \tag{3.10}$$

where

$$\alpha_p = \sum_{l=0}^{N} \int_0^4 \left(\frac{s}{4-s}\right)^{1/2} \phi_{l,p}(s)\, \mathrm{d}s \qquad p = 0, 1, \ldots. \tag{3.11}$$

As already mentioned, we will be interested in correlating values of partial waves at unphysical energies, particularly on the real axis $0 < s < 4$. Therefore it is useful to recall at this point some known rigorous results concerning the behaviour of partial waves at particular points in this region. We shall refer here particularly to the S and D waves, for which a collection of rather impressive constraints has been obtained by various authors (Martin 1967, Common 1968, Grassberger 1973, Diţă 1973). For instance, the S wave $a_0(s)$ is known to be a convex function on $0 \leqslant s \leqslant 1{\cdot}7$, having a minimum in the interval $1{\cdot}219 \leqslant s \leqslant 1{\cdot}697$. Its values at some couples of real points satisfy the inequalities found by Martin (1967), namely

$$\begin{aligned} &a_0(0{\cdot}2133) \leqslant a_0(3{\cdot}2029)\\ &a_0(0{\cdot}1489) \leqslant a_0(3{\cdot}2949)\\ &a_0(0{\cdot}3092) \leqslant a_0(3{\cdot}0826)\\ &a_0(s) < a_0(4), \qquad \forall s \in (0, 4). \end{aligned} \tag{3.12}$$

As concerns the D wave, it was shown to be non-negative on $0 < s < 4$ and decreasing on the interval $1{\cdot}434 \leqslant s \leqslant 4$.

Besides these known properties, we noticed that, starting from some bounds on the total amplitude $F(s, t)$ (Lopez and Mennessier 1977), we could derive, by a simple method, numerical bounds on the partial waves at some particular points. The method and some results concerning the S and D waves, which were of interest in our applications, are presented in appendix 2.

We shall now indicate how the study of $\pi\pi$ amplitudes can be brought to the canonical form of the preceding section. This can be accomplished through the following steps.

First we perform the conformal transformation (Caprini and Diţă 1978)

$$z(s) = \frac{12s + \mathrm{i}[13(s-4)(11s+52)]^{1/2}}{12s - \mathrm{i}[13(s-4)(11s+52)]^{1/2}} \tag{3.13}$$

which maps the domain (3.3) onto the unit disc $|z| < 1$, the cut $4 \leqslant s \leqslant 52$ being applied on the right semicircle and the real segment $0 \leqslant s \leqslant 4$ becoming the diameter $-1 \leqslant z \leqslant 1$.

We define then the real analytic outer functions

$$G_l(z) = \exp\left(\frac{1-z^2}{\pi} \int_{\pi/2}^{\pi} \frac{\ln M_l(\theta)\, \mathrm{d}\theta}{1 - 2z\cos\theta + z^2}\right). \tag{3.14}$$

By construction $G_l(z)$ have modulus equal to 1 on the right semicircle and $M_l(\theta) \equiv M_l(s(\theta))$ on the left one and have no zeros inside the unit disc $|z| < 1$. In (3.14) we took into account the equality $M_l(\theta) = M_l(-\theta)$ which follows from the property of $S_l(z)$ of being real analytic, i.e. $S_l(\bar{z}) = \overline{S_l(z)}$. It is also of interest to point out here that $G_l(z)$ behave like $\ln s \approx \ln(1+z)^{1/2}$ around $s = 0$, i.e. $z = -1$, and are finite at $s = 4$, i.e. $z = 1$, as follows from the behaviour of $M_l(\theta)$ (Bonnier 1975).

We consider now the functions

$$f_l(z) = S_l(z)/G_l(z). \tag{3.15}$$

They are real analytic inside the domain $|z|<1$ and satisfy the boundedness condition

$$|f_l(z)| \leq 1 \qquad |z|<1 \tag{3.16}$$

i.e. they belong to the unit sphere of H^∞. This follows from (3.4), (3.5) and (3.14). Let us next introduce the functions $f_l(z)$ in the crossing integrals (3.10) and split for convenience the sum in terms containing only one function f_l. We obtain a set of relations of the form

$$\int_{-1}^{1} \psi_{l,p}(x) f_l(x)\,\mathrm{d}x = \xi_{l,p}. \tag{3.17}$$

Here the functions $\psi_{l,p}(x)$ are defined as

$$\psi_{l,p}(x) = \phi_{l,p}(x)\left(\frac{s(x)}{4-s(x)}\right)^{1/2} G_l(x)\,\mathrm{d}s(x)/\mathrm{d}x \tag{3.18}$$

with $s(x)$ and $\mathrm{d}s/\mathrm{d}x$ being the inverse of (3.13) and the Jacobian of the transformation, respectively:

$$\begin{aligned} s(x) &= \frac{52(1+x)}{1+x+12[2(1+x^2)]^{1/2}} \\ \frac{\mathrm{d}s}{\mathrm{d}x} &= \frac{624\sqrt{2}(1-x)}{(1+x^2)^{1/2}\{1+x+12[2(1+x^2)]^{1/2}\}}. \end{aligned} \tag{3.19}$$

Moreover, from (3.10) it follows that the numbers $\xi_{l,p}$ satisfy the relations

$$\sum_{l=0}^{M} \xi_{l,p} = \alpha_p \qquad p = 0, 1, \ldots \tag{3.20}$$

with α_p defined in (3.11). For instance, after some simple calculations the relations (3.4) for the S and D waves become

$$\begin{aligned} &\int_{-1}^{1} \psi_{0,0}(x) f_0(x)\,\mathrm{d}x = 4\pi \\ &\psi_{0,0}(x) = [s(4-s)]^{1/2}(3s-4)G_0(x)\,\mathrm{d}s/\mathrm{d}x \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\int_{-1}^{1} \psi_{0,1}(x) f_0(x)\,\mathrm{d}x = \xi_{0,1} \\ &\int_{-1}^{1} \psi_{2,1}(x) f_2(x)\,\mathrm{d}x = \xi_{0,2} \\ &\xi_{0,1} + \xi_{0,2} = 8\pi \\ &\psi_{0,1}(x) = 4(4-s)(s-1)G_0(x)\,\mathrm{d}s/\mathrm{d}x \\ &\psi_{2,1}(x) = (4-s)^2[s(4-s)]^{1/2}G_2(x)\,\mathrm{d}s/\mathrm{d}x. \end{aligned} \tag{3.22}$$

Actually, the equations (3.16) and (3.17) express the constraints on the $\pi\pi$ partial waves in the form considered at the beginning of § 2. Suppose first that we are interested in finding the range of values of one partial wave $a_l(s)$ at some particular points s_i, $i = 1, \ldots, n$, compatible with the above constraints. The method developed in the previous section describes the domain $\mathscr{D}_g$ of values of the function

$$f_l(z)=\left[1-2\left(\frac{4-s}{s}\right)^{1/2}a_l(z)\right]G_l(z)^{-1}$$

at $z_i=z(s_i)$. In particular, for every admissible function $g(z)$, the inequality (2.36) describing the domain $\mathscr{D}_g$ yields an explicit consistency condition among the values $f_i\equiv f_l(z_i)$ and therefore among $a_l(s_i)$. From these relations we can use the information available on $a_l(s)$ at some points in order to constrain $a_l(s)$ at other points. To this end we notice that the norm appearing in (2.36) is a convex function of $\{f_j\}$. Therefore, the extremum values of one parameter f_i, the others being kept fixed, are obtained by taking the equality sign in (2.36) and solving with respect to the corresponding f_i the equation thus obtained. As this equation is quadratic in f_i, we obtain from it explicit expressions for the upper and lower bounds upon one f_i, in terms of the remaining values $\{f_j\}_{j\neq i}$. Now, if we restrict ourselves to $a_0(s)$ and want to take into account the constraints (3.12) and (A2.2), we must introduce in the set $\{s_i\}$, besides the point s_0 at which bounds are sought, the relevant points for these constraints. Clearly, when expressed in terms of $f_0(z)$ the above constraints remain simple linear inequalities among the f_i. The problem of bounding the amplitude at interior points was thus reduced to finding the extrema of an explicit expression of f_i, the parameters being subject to a finite number of linear constraints.

With this method we derived upper and lower bounds on the S and D waves at some unphysical energies. In our numerical applications we have treated the partial waves only separately so far, i.e. with no crossing relations of the form (3.8) taken into account, and introduced into the problem a rather small number of interior constraints. A more complete numerical application of the formalism raises no special difficulties, requiring only an increased computational time.

Some of the results obtained in this way are, however, interesting. For instance, we studied the problem of a lower bound on the S-wave scattering length $a^{00}=a_0(4)$. Actually $s=4$ is a point on the frontier and cannot be properly included in our formalism. Nevertheless, having in mind the last Martin inequality (3.12) we looked for a lower bound upon $a_0(s_0)$, the point s_0 being close to $s=4$, this yielding a lower bound for a^{00} too. If only the constraints (3.4) and (3.5) are considered, the maximum modulus principle applied for $f_0(z)=S_0(z)/G_0(z)$ gives†

$$a^{00}>-\frac{\sqrt{78}}{12\pi}\int_{\pi/2}^{\pi}\frac{\ln M_0(\theta)}{\sin^2\theta/2}\,\mathrm{d}\theta=-2{\cdot}66.$$

By making use of the additional constraints (3.12) and (A2.2) and of the convexity of $a_0(s)$, we could improve this value up to $a^{00}>-1{\cdot}7$ which practically coincides with the best result known at present (Lopez and Mennessier 1977) mentioned in the introduction. Our analysis shows that a more complete use of the rigorous constraints improves our knowledge of the amplitude values and suggests that, proceeding along this line, a further improvement of the present lower bound is possible.

With the same method we also calculated new bounds for S and D waves at interior points. Several results concerning the S wave were reported previously (Caprini and Diţă 1978). Below we give, for illustration, some of our new results for the D wave. By

† This value is slightly better than the bound $a^{00}>-3{\cdot}3$ obtained with the same method by Bonnier (1975), since we used, for computing $M_0(\theta)$ from the Bonnier formula, better bounds on $F(2, 2)$, $F(3, 2)$ and $F(\frac{4}{3}, \frac{4}{3})$ (Lopez and Mennessier 1977).

making use of the bounds (A2.7) derived by us in appendix 2 we obtained

$$a_2(0{\cdot}1) < 28{\cdot}55$$

$$a_2(0{\cdot}4) < 10{\cdot}4$$

$$a_2(1{\cdot}2) \leq 2{\cdot}22$$

$$a_2(1{\cdot}5) \leq 1{\cdot}51.$$

As we mentioned earlier, the most interesting feature of our method is its ability to connect the admissible domains of values taken by different partial waves. This connection is established through the integral relations (3.8) which are treated exactly in our approach. Practically, the joined domain of values for a given number of partial waves is obtained by intersecting with the hyperplanes (3.20) the separate domains for each partial wave, these depending explicitly on the parameters $\xi_{l,p}$ as may be seen from (2.36). In this way the content of crossing symmetry, expressed in the integral relations (3.8), can be more fully taken into account.

The above description gives immediately new rigorous correlations among values of different partial waves at coincident or different energies. For instance, if $S_0(s_0)$ and $S_2(s_2)$ are the S-matrix elements for the S and D waves taken at the arbitrary points s_0 and s_2 respectively, we can write down the simple inequalities

$$\begin{aligned} 18\pi &\leq \sum_{l=0,2} [\alpha_l S_l(\alpha_l)/G_l(s_l) + \beta_l(1 - S_l^2(s_l)/G_l^2(s_l))^{1/2}] \\ \sum_{l=0,2} a_l S_l(\alpha_l)/G_l(s_l) &\leq 18\pi + \sum_{l=0,2} \beta_l(1 - S_l^2(s_l)/G_l^2(s_l))^{1/2} \end{aligned} \tag{3.23}$$

where

$$\alpha_l = \int_{-1}^{1} \psi_{l,1}(x)\,\mathrm{d}x$$

$$\beta_l = \frac{1}{(1-x_l^2)^{1/2}} \int_{-1}^{1}\int_{-1}^{1} \frac{\psi_{l,1}(x)\psi_{l,2}(y)(x-x_l)(y-y_l)\,\mathrm{d}x\,\mathrm{d}y}{1-xy}$$

and the functions $G_l(x)$ and $\psi_{l,i}(x)$ are defined in (3.14) and (3.22), respectively.

We obtained these relations by writing for both the S and D waves the inequality (2.36) with $n = 1$, $p = 1$ and $g(z)$ taken for simplicity of the form (2.37) with $\alpha = x_l$, and subsequently eliminating through (3.22) the parameters $\xi_{l,i}$ entering the integral conditions. From this procedure it follows that the relations (3.23) are not the optimal ones: an optimisation upon $g(z)$ actually gives the sharpest inequality connecting the values $S_0(s_0)$ and $S_2(s_2)$.

From these relations one can obtain rigorous correlations between the positions of resonances in various partial waves. Indeed, if we recall that the poles of $S_l(z)$ on the second Riemann sheet induce zeros of $S_l(z)$ on the first sheet (Bonnier and Donohue 1978), we have to set $S_l(s_l) = 0$ in the inequalities of the type (3.23) (actually for complex s_l), being left with rigorous correlations among the points s_l where resonances of spin l can be located.

With the admissible domain of values described above, the problem of computing bounds on a particular partial wave by taking into account the influence of the others is in principle solved. We think that in this way it would be possible to answer the open question of saturating the lower bounds on the S wave. Actually, our formalism also yields bounds for the total amplitude $F(s, t)$. Indeed, as results from the relation (A2.9) of appendix 2, $F(s, t)$ can be bounded by two linear expressions involving a finite number of partial waves, in the form

$$\sum_{l=0}^{N} (2l+1)a_l(s)P_l\left(1+\frac{2t}{s-4}\right) + I_m(s, t)a_n(s)$$
$$\leqslant F(s, t) \leqslant \sum_{l=0}^{N} (2l+1)a_l(s)P_l\left(1+\frac{2t}{s-4}\right) + I_M(s, t)a_n(s) \qquad n \geqslant 2 \quad (3.24)$$

where a_n can be any partial wave $n \geqslant 2$, $I_m(s, t)$ and $I_M(s, t)$ being known functions. Knowing explicitly the admissible domain for the partial waves $a_l(s)$, the problem of bounding $F(s, t)$ amounts then to a standard numerical optimisation. An application of particular interest would be to investigate the special values $F(2, 2)$, $F(3, 2)$ and $F(\frac{4}{3}, \frac{4}{3})$, which were used as input in the computation of the Bonnier bound (3.5) and in the particular bounds of appendix 2. By taking a large enough number N of partial waves in (3.24), in order to exploit to a large extent the crossing symmetry contained in the integral constraints (3.8), the output values of the above $F(s, t)$ must become better than the input ones or at least will reproduce them, if an ultimate limit has indeed been reached. The numerical investigation of such a problem is a project for future work.

Appendix 1

We shall give in this appendix the proof of the equality $\mathscr{D} = \bigcap_g \mathscr{D}_g$. The intersection is taken here upon the set, denoted in what follows by $\bar{S}_2$, of the outer analytic functions $g(z)$, described by the relations (2.3) and (2.4) of § 2.

Using the definitions of the domains $\mathscr{D}$ and $\mathscr{D}_g$ given in § 2 and having in mind the remark leading to relation (2.5), it is easy to see that the problem which must be solved is to prove the equality

$$\mu_\infty(\{f_i\}, \{\alpha_K\}) = \sup_{g \in \bar{S}_2} \mu_2(g, \{f_i\}, \{\alpha_K\}) \qquad \text{(A1.1)}$$

where $\mu_2(g, \{f_i\}, \{\alpha_K\})$ is the minimal L^2 norm defined in (2.5) with the constraints (2.6) and $\mu_\infty(\{f_i\}, \{\alpha_K\})$ denotes the minimal L^∞ norm

$$\mu_\infty(\{f_i\}, \{\alpha_K\}) = \inf \|f\|_\infty \qquad \text{(A1.2)}$$

taken with the same constraints.

We notice first that for an arbitrary function $f \in H^\infty$ we can write the inequality $\|f\|_\infty \geqslant \|gf\|_2$, which holds for every $g \in \bar{S}_2$, and consequently we have

$$\|f\|_\infty \geqslant \sup_{g \in \bar{S}_2} \|gf\|_2.$$

As a matter of fact this inequality is saturated, as can easily be verified by the explicit construction of a suitable $g(z)$. If we now take the infimum of both norms for functions $f(z)$ subject to the constraints (2.6) and take into account the fact that inf sup $\geqslant$ sup inf, we obtain the inequality

$$\mu_\infty(\{f_i\}, \{\alpha_K\}) \geqslant \sup_{g \in \bar{S}_2} \mu_2(g, \{f_i\}, \{\alpha_K\}). \qquad \text{(A1.3)}$$

We have therefore to prove the saturation of this inequality. To this end we shall resort again to the duality theorem (Duren 1970, Luenberger 1968) for both the H^∞ and H^2 minimal norms. Using the same technique as in § 2, we obtain finally from (A1.3) the equivalent inequality

$$\sup_{\substack{G\in L^1\\ \|G\|_1\le 1}}\left|\frac{1}{2\pi i}\oint G(\zeta)k(\zeta)\,d\zeta-\sup_{h\in\mathcal{H}}\frac{1}{2\pi i}\oint G(\zeta)h(\zeta)\,d\zeta\right|$$

$$\geqslant \sup_{g\in S_2}\sup_{\substack{F\in L^2\\ \|F\|_2\le 1}}\left|\frac{1}{2\pi i}\oint F(\zeta)g(\zeta)k(\zeta)\,d\zeta-\sup_{h\in\mathcal{H}}\frac{1}{2\pi i}\oint F(\zeta)g(\zeta)h(\zeta)\,d\zeta\right|. \qquad (A1.4)$$

We recall that $k(\zeta)$ is the rational function defined in (2.11), $G(\zeta)$ and $F(\zeta)$ are arbitrary functions from the unit spheres of L^1 and L^2, respectively, while the convex set $\mathcal{H}$ to which $h(\zeta)$ belongs was defined in § 2. For convenience we shall rewrite here the integral conditions (2.9) defining $\mathcal{H}$ in the form

$$\sum_{j=0}^{\infty} h_j d_K^j=\beta_K \qquad K=1,\ldots,p \qquad (A1.5)$$

where h_j are the Fourier coefficients of the analytic function $h\in H^\infty$, i.e.

$$h(\zeta)=\sum_{j=0}^{\infty} h_j\zeta^j \qquad |\zeta|\le 1 \qquad (A1.6)$$

and

$$d_K^j=\int_{-1}^{1}\psi_K(x)B_{n+1}(x)x^j\,dx \qquad K=1,\ldots,p. \qquad (A1.7)$$

It is useful to recall also from § 2 that the structure of the support functional in (A1.4) restricts the form of the functions $G(\zeta)$ which are relevant for the supremum to

$$G(\zeta)=G_+(\zeta)+\sum_{j=1}^{p} G_{-j}P_j(\zeta)/\zeta^j \qquad (A1.8)$$

where $G_+(\zeta)$ is an arbitrary analytic function belonging to H^1 and the functions $P_j(\zeta)$ are defined by

$$P_j(\zeta)=1+\frac{(-1)^{j+1}}{\zeta^{p-j}\bar{\Delta}}\times\det\begin{vmatrix}\displaystyle\int_{-1}^{1}\frac{\psi_1(x)B_{n+1}(x)x^p}{\zeta-x}\,dx & d_1^0 & \ldots & d_1^{j-1} & d_1^{j+1} & \ldots & d_1^{p-1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \displaystyle\int_{-1}^{1}\frac{\psi_p(x)B_{n+1}(x)x^p}{\zeta-x}\,dx & d_p^0 & \ldots & d_p^{j-1} & d_p^{j+1} & \ldots & d_p^{p-1}\end{vmatrix} \qquad (A1.9)$$

where

$$\bar{\Delta}=\det\begin{vmatrix} d_1^0 & \ldots & d_1^{p-1}\\ \vdots & & \vdots\\ d_p^0 & \ldots & d_p^{p-1}\end{vmatrix}.$$

These expressions are similar to equations (2.20) and (2.21) of § 2.

In order to prove the saturation of (A1.4) we shall proceed in two steps. First, let us consider a truncated problem, obtained by replacing the convex set $\mathscr{H}$ defined in (A1.5) by the convex $\mathscr{H}_N$, consisting of functions $h(\zeta)$ obeying the conditions

$$\sum_{j=0}^{N} h_j d_K^j = \beta_K \qquad K = 1, \ldots, p, N > p. \tag{A1.10}$$

By comparing (A1.10) with (A1.5) one can see that this is equivalent to setting the higher moments d_K^j, $j \geq N$ of $\psi_K(x)$ equal to zero. Denoting by $\mu_\infty^{(N)}$ and $\mu_2^{(N)}$ the minimal norms corresponding to the convex $\mathscr{H}_N$, we shall first prove the equality

$$\mu_\infty^{(N)}(\{f_i\}, \{\alpha_K\}) = \sup_{g \in S_2} \mu_2^{(N)}(g, \{f_i\}, \{\alpha_K\}). \tag{A1.11}$$

By applying the duality theorem this amounts to showing the saturation of an inequality similar to (A1.4), namely

$$\sup_{\substack{G \in L^1 \\ \|G\|_1 \leq 1}} \left| \frac{1}{2\pi i} \oint G(\zeta) k(\zeta)\, d\zeta - \sup_{h \in \mathscr{H}_N} \frac{1}{2\pi i} \oint G(\zeta) h(\zeta)\, d\zeta \right|$$

$$\geq \sup_{g \in S_2} \sup_{\substack{F \in L^2 \\ \|F\|_2 \leq 1}} \left| \frac{1}{2\pi i} \oint F(\zeta) g(\zeta) k(\zeta)\, d\zeta - \sup_{h \in \mathscr{H}_N} \frac{1}{2\pi i} \oint F(\zeta) g(\zeta) h(\zeta)\, d\zeta \right|. \tag{A1.4$'$}$$

Let us denote by $G_0^{(N)}(\zeta)$ the extremal function for the supremum on the left-hand side of (A1.4′), which is known to exist (Duren 1970). From the arguments preceding (A1.9) it follows that $G_0^{(N)}$ will have the form

$$G_0^{(N)}(\zeta) = G_+(\zeta) + \sum_{j=1}^{p} G_{-j} P_j^{(N)}(\zeta)/\zeta^j \tag{A1.12}$$

where $P_j^{(N)}(\zeta)$ are now polynomials in $1/\zeta$, of degree equal to $N-j$, defined by

$$P_j^{(N)}(\zeta) = 1 + \frac{(-1)^{j+1}}{\zeta^{p-j}\bar{\Delta}}$$

$$\times \det \begin{vmatrix} \int_{-1}^{1} \psi_1(x) B_{n+1}(x) x^p \sum_{K=1}^{N-p-1} \frac{x^K}{\zeta^{K+1}}\, dx & d_1^0 & \ldots & d_1^{j-1} & d_1^{j+1} & \ldots & d_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \int_{-1}^{1} \psi_p(x) B_{n+1}(x) x^p \sum_{K=1}^{N-p-1} \frac{x^K}{\zeta^{K+1}}\, dx & d_p^0 & \ldots & d_p^{j-1} & d_p^{j+1} & \ldots & d_p^{p-1} \end{vmatrix}. \tag{A1.13}$$

As follows from (A1.12) and (A1.13) the lowest negative frequency of $G_0^{(N)}(\zeta)$ is ζ^{-N}. By multiplying (A1.12) by ζ^N we hence obtain an analytic function belonging to H^1. Let us write for this function the Riesz factorisation (Duren 1970):

$$\zeta^N G_0^{(N)}(\zeta) = B(\zeta) \phi(\zeta) \tag{A1.14}$$

where $B(\zeta)$ is the inner factor and $\phi(\zeta)$ is the outer one. We can now define two functions $g_0^{(N)}(\zeta)$ and $F_0^{(N)}(\zeta)$ by the relations

$$g_0^{(N)}(\zeta)=(\phi(\zeta))^{1/2}$$
$$F_0^{(N)}(\zeta)=B(\zeta)(\phi(\zeta))^{1/2}/\zeta^N \tag{A1.15}$$

which implies that

$$G_0^{(N)}(\zeta)=g_0^{(N)}(\zeta)F_0^{(N)}(\zeta). \tag{A1.16}$$

Using the fact that by definition $\|G_0^{(N)}\|_1\leqslant 1$, we see that $g_0^{(N)}(\zeta)$ is an outer function with $\|g_0^{(N)}\|\leqslant 1$ while $F_0^{(N)}(\zeta)$ belongs to L^2 and $\|F_0^{(N)}\|_2\leqslant 1$. This means that $g_0^{(N)}(\zeta)$ and $F_0^{(N)}(\zeta)$ satisfy the properties required for $g(\zeta)$ and $F(\zeta)$ in the right-hand side of (A1.4′), and moreover, due to (A1.16), they saturate this inequality. The relation (A1.11) was thus proved. We point out that the above argument was based on the factorisation of $G_0^{(N)}(\zeta)$, which was possible due to the particular form (A1.12) of its non-analytic part, specific for the truncated problem. A factorisation of a function $G(\zeta)$ having the form (A1.8) is not known.

As the equality (A1.9) holds for every N, it will apply also for the limits of both sides when $N\to\infty$, if they exist. We pass therefore to the second step of our proof, which consists in verifying the equality:

$$\lim_{N\to\infty}\mu_K^{(N)}(\{f_i\},\{\alpha_K\})=\mu_K(\{f_i\},\{\alpha_K\}) \qquad \text{for } K=2,\infty. \tag{A1.17}$$

In what follows we shall treat explicitly the L^∞ norm, the proof for the L^2 case being similar. We start by noting that, using the expressions (A1.8) and (A1.12) of the optimal functions in (A1.4) and (A1.4′), respectively, we can write $\mu_\infty^{(N)}$ and μ_∞ in the form

$$\mu_\infty^{(N)}(\{f_i\},\{\alpha_K\})=\sup_{\substack{\|G\|_1\leqslant 1\\ G(\zeta)=G_+(\zeta)+\sum_{j=1}^{p}G_{-j}P_j^{(N)}(\zeta)/\zeta^j}}|\Phi(g)| \tag{A1.18}$$

$$\mu_\infty(\{f_i\},\{\alpha_K\})=\sup_{\substack{\|G\|_1\leqslant 1\\ G(\zeta)=G_+(\zeta)+\sum_{j=1}^{p}G_{-j}P_j(\zeta)/\zeta^j}}|\Phi(G)| \tag{A1.19}$$

where $\Phi(G)$ is the linear functional

$$\Phi(G)=\frac{1}{2\pi\mathrm{i}}\oint G_+(\zeta)k(\zeta)\,\mathrm{d}\zeta+\sum_{j=1}^{p}G_{-j}\xi_j \tag{A1.20}$$

and the numbers ξ_j are defined, similarly to (2.27), as

$$\xi_j=\frac{(-1)^j}{\bar{\Delta}}\det\begin{vmatrix}\beta_1 & d_1^0 & \dots & d_1^{j-1} & d_1^{j+1} & \dots & d_1^{p-1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \beta_p & d_p^0 & \dots & d_p^{j-1} & d_p^{j+1} & \dots & d_p^{p-1}\end{vmatrix}.$$

In order to show the equality of (A1.18) and (A1.19) we shall use the following trick: let us take $\Phi(G)$ fixed at a value κ and consider the minimum norm problems:

$$\mathscr{F}^{(N)}(\kappa)=\inf_{\Phi(G)=\kappa}\left\|G_+(\zeta)+\sum_{j=1}^{p}G_{-j}P_j^{(N)}(\zeta)/\zeta^j\right\|_1 \tag{A1.21}$$

$$\mathscr{F}(\kappa)=\inf_{\Phi(G)=\kappa}\left\|G_+(\zeta)+\sum_{j=1}^{p}{}_{,}G_jP_j(\zeta)/\zeta^j\right\|_1. \tag{A1.22}$$

It is not difficult to see that the minimal norms (A1.21) and (A1.22) are convex functions of κ. This implies that $\mu_\infty^{(N)}$ and μ_∞ are equal to the largest solutions of the equations

$$\mathscr{F}^{(N)}(\kappa)=1 \tag{A1.23}$$

and

$$\mathscr{F}(\kappa)=1 \tag{A1.23'}$$

respectively. In order to prove the relation (A1.14) it is therefore sufficient to show that

$$\lim_{N\to\infty} \mathscr{F}^{(N)}(\kappa)=\mathscr{F}(\kappa). \tag{A1.24}$$

The problem was thus reduced to the study of the minimal norms (A1.21) and (A1.22) taken upon the analytic functions $G_+(\zeta)\in H^1$ and p parameters G_{-j}, constrained by the linear relation $\Phi(G)=\kappa$. As will become clear below, for our purpose we have only to consider the minimisation upon $G_+(\zeta)$, with the parameters G_{-j} kept fixed. More precisely, we have to prove, at fixed κ and G_{-j}, the equality

$$\lim_{N\to\infty} \inf_{G_+\in\mathscr{H}} \|G_+-\chi^{(N)}\|_1 = \inf_{G_+\in\mathscr{H}} \|G_+-\chi\|_1 \tag{A1.25}$$

where

$$\begin{aligned} \chi^N(\zeta) &= -\sum_{j=1}^{p} G_{-j}P_j^{(N)}(\zeta)/\zeta^j \\ \chi(\zeta) &= -\sum_{j=1}^{p} G_{-j}P_j(\zeta)/\zeta^j \end{aligned} \tag{A1.26}$$

the infimum being taken upon the analytic functions $G_+(\zeta)$ belonging to the hyperplane

$$\mathscr{H}=\left\{G_+(\zeta)\,\middle|\, G_+\in H^1, \frac{1}{2\pi i}\oint G_+(\zeta)k(\zeta)\,d\zeta=\kappa-\sum_{j=1}^{p} G_{-j}\xi_j=\text{constant}\right\}. \tag{A1.27}$$

Indeed, if (A1.25) is true for every set of parameters G_{-j}, it will hold also for the minimum with respect to G_{-j}, and this gives the desired result (A1.24).

In (A1.25) one may recognise the distances from the functions $\chi^{(N)}$ and χ to the hyperplane $\mathscr{H}$. We shall first show that

$$\lim_{N\to\infty} \|\chi^{(N)}-\chi\|_1=0. \tag{A1.28}$$

Using the expressions (A1.9) and (A1.13) in (A1.26) we evaluate the difference

$$\begin{aligned} \chi^N(\zeta)-\chi(\zeta) &= \sum_{j=1}^{p} G_{-j}(P_j^{(N)}(\zeta)-P_j(\zeta))/\zeta^j \\ &= \sum_{j=1}^{p} \frac{(-1)^{j-1}G_{-j}}{\zeta^p\bar{\Delta}} \\ &\quad\times\det\begin{vmatrix} \int_{-1}^{1} \psi_1(x)B_{n+1}(x)x^p \sum_{K=N-p}^{\infty}\frac{x^K}{\zeta^{K+1}}\,dx & d_1^0 & \dots & d_1^{j-1} & d_1^{j+1} & \dots & d_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \int_{-1}^{1} \psi_p(x)B_{n+1}(x)x^p \sum_{K=N-p}^{\infty}\frac{x^K}{\zeta^{K+1}}\,dx & d_p^0 & \dots & d_p^{j-1} & d_p^{j+1} & \dots & d_p^{p-1} \end{vmatrix} \end{aligned}$$

$$= \sum_{j=1}^{p} \frac{(-1)^{j-1} G_{-j}}{\zeta^N \bar{\Delta}}$$

$$\times \det \begin{vmatrix} \displaystyle\int_{-1}^{1} \frac{\psi_1(x) x^N B_{n+1}(x)\,\mathrm{d}x}{\zeta - x} & d_1^0 & \ldots & d_1^{j-1} & d_1^{j+1} & \ldots & d_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \displaystyle\int_{-1}^{1} \frac{\psi_p(x) x^N B_{n+1}(x)\,\mathrm{d}x}{\zeta - x} & d_p^0 & \ldots & d_p^{j-1} & d_p^{j+1} & \ldots & d_p^{p-1} \end{vmatrix}. \tag{A1.29}$$

Using this expression in the definition of the L^1 norm appearing in (A1.28) and changing the order of integration with respect to $\mathrm{d}x$ and $|\mathrm{d}\zeta|$, we obtain after some simple calculations the upper bound

$$\|\chi^{(N)} - \chi\|_1 = \frac{1}{2\pi} \oint |\chi^{(N)}(\zeta) - \chi(\zeta)|\, |\mathrm{d}\zeta| \leq \int_{-1}^{1} |\bar{\psi}(x) x^N| \mathrm{d}x \tag{A1.30}$$

where

$$\overline{\psi(x)} = \sum_{j=1}^{p} \gamma_j \psi_j(x) B_{n+1}(x) \oint \frac{|\mathrm{d}\zeta|}{|\zeta - x|} \tag{A1.31}$$

γ_j being some finite coefficients which result from (A1.29). One can verify immediately that $\overline{\psi(x)}$ is bounded on $[-1, 1]$. This is evident for $|x| < 1$, so that the only points that need special study are $x = \pm 1$. As follows from § 3, all the functions $\psi_j(x)$ contain a factor $(4-s)^{1/2} \approx (1-x)$ from the crossing functions, and another $(1-x)$ from the Jacobian $\mathrm{d}s/\mathrm{d}x$ of the conformal transformation (3.19), which cancel the logarithmic singularity of the integral over $|\mathrm{d}\zeta|$ at $x = 1$, so that at this point $\bar{\psi}_j(x)$ vanishes. On the other hand, at $x = -1$ the functions $\psi_j(x)$ behave like $\sqrt{s} \ln s \approx (1+x) \ln(1+x)$, the first factor coming from the crossing functions and the logarithm from the outer function $G(x)$ defined in (3.14). These factors, together with the factor $\ln(1+x)$ yielded by the integral over $|\mathrm{d}\zeta|$, make $\psi(x)$ to be still finite at $x = 1$. If we denote now by M the maximum of $\bar{\psi}(x)$ over $[-1, 1]$, we obtain from (A1.30)

$$\|\chi^{(N)} - \chi\|_1 \leq M \int_{-1}^{1} |x^N|\, \mathrm{d}x = 2M/(N+1)$$

which gives, for $N \to \infty$, the desired result (A1.28). By making use of this result we can immediately prove the equality (A1.25). Indeed, let us denote by $G_{+,0}^{(N)}(\zeta)$ and $G_{+,0}(\zeta)$ the functions achieving the minimum in the left- and right-hand sides of this relation. The existence of these functions can easily be established using the general theory of extremal problems (Duren 1970). We can then write

$$\inf_{G_+ \in \mathcal{H}} \|G_+ - \chi\|_1 \leq \|G_{+,0}^{(N)} - \chi\|_1 \leq \|G_{+,0}^{(N)} - \chi^{(N)}\|_1 + \|\chi^{(N)} - \chi\|_1$$

$$\inf_{G_+ \in \mathcal{H}} \|G_+ - \chi^{(N)}\|_1 \leq \|G_{+,0} - \chi^{(N)}\|_1 \leq \|G_{+,0} - \chi\|_1 + \|\chi^{(N)} - \chi\|_1.$$

By now taking the limit $N \to \infty$ of both the above relations and using (A1.28) we obtain two opposite inequalities

$$\inf_{G_+ \in \mathcal{H}} \|G_+ - \chi\|_1 \leq \lim_{N \to \infty} \inf_{G_+ \in \mathcal{H}} \|G_+ - \chi^{(N)}\|_1$$

$$\lim_{N \to \infty} \inf_{G_+ \in \mathcal{H}} \|G_+ - \chi^{(N)}\|_1 \leq \inf_{G_+ \in \mathcal{H}} \|G_+ - \chi\|_1$$

which terminate the proof of the equality (A1.25). Going back, this implies equation (A1.24) and consequently the desired result (A1.17). One can easily verify that the above arguments can be applied without significant modifications to the L^2 norm, for a given $g \in \bar{S}_2$. The relation (A1.17), together with the preceding one (A1.11), terminate the proof of (A1.1) and hence of the equality $\mathscr{D} = \bigcap_g \mathscr{D}_g$.

Appendix 2

In this appendix we shall present a simple method for obtaining numerical bounds on the partial waves $a_l(s)$ for $0<s<4$, using as input the numerical information on the $\pi^0\pi^0$ total amplitude $F(s, t)$. The technique is inspired from our previous work (Diţă 1973) and some results concerning the S wave were reported elsewhere (Caprini and Diţă 1978).

The starting point in our derivation is a once-subtracted fixed-t dispersion relation, with the subtraction term expressed in functions of the S wave:

$$F(s, t) = a_0(t) + \frac{1}{\pi}\int_4^\infty A(x, t)K_0(x, s, t)\, dx \tag{A2.1}$$

where

$$K_0(x, s, t) = \frac{1}{x-s} + \frac{1}{x-u} + \frac{2}{4-t}\ln\left(1 + \frac{t-4}{x}\right).$$

From (A2.1), by taking into account the positivity property of the absorptive part $A(x, t)$ for $x \geqslant 4$ and $0 \leqslant t \leqslant 4$, we can bound $a_0(t)$ in terms of the total amplitude $F(s, t)$ at those values of s and t where the kernel $K_0(x, s, t)$ has a constant sign in the integration range $4 \leqslant x < \infty$. For instance, by noticing that $K_0(x, \frac{4}{3}, \frac{4}{3}) \leqslant 0$ for $x \geqslant 4$ we obtained

$$a_0(\tfrac{4}{3}) > F(\tfrac{4}{3}, \tfrac{4}{3}) > -8{\cdot}2$$

where we used the lower bound on $F(\frac{4}{3}, \frac{4}{3})$ obtained by Lopez and Mennessier (1977).

In a similar way the following bounds can be immediately derived:

$$\begin{aligned}
&-7{\cdot}25 < F(2, 1) < a_0(1) < F(3, 1) < 3{\cdot}2\\
&-8{\cdot}2 < F(\tfrac{4}{3}, \tfrac{4}{3}) < a_0(\tfrac{4}{3}) < F(0, \tfrac{4}{3}) < 3{\cdot}05\\
&-7{\cdot}25 < F(1, 2) < a_0(2) < F(2, 2) < 2{\cdot}9\\
&-3{\cdot}3 < F(3, 0) < a_0(0)\\
&a_0(\tfrac{8}{3}) < F(0, \tfrac{8}{3}) < 3{\cdot}05\\
&a_0(3) < F(1, 3) < 3{\cdot}2.
\end{aligned} \tag{A2.2}$$

In order to find bounds on the D wave we take the difference of two relations of the form (A2.1) written for the same t and different values of s:

$$F(s_1, t, u_1) - F(s_2, t, u_2) = \frac{(s_1-s_2)(s_1-u_2)}{\pi}\int_4^\infty A(x, t)\frac{(2x+t-4)\, dx}{(x-s_1)(x-u_1)(x-s_2)(x-u_2)}. \tag{A2.3}$$

The Froissart–Gribov representation for the partial wave $a_l(t)$

$$a_l(t)=\frac{4}{\pi(4-t)}\int_4^\infty A(x,t)Q_l\left(\frac{2x}{4-t}-1\right)\,\mathrm{d}x$$

valid for $l\geqslant 2$, together with the positivity of the absorptive part $A(x,t)$ and of the Legendre functions Q_l, imply the positivity of the partial waves:

$$a_l(t)>0 \qquad 0\leqslant t<4,\ l\geqslant 2.$$

This last property allows us to define a function with bounded variation on $(4,\infty)$, as follows:

$$\psi_t(x)=\frac{4}{\pi(4-t)a_2(t)}\int_4^x A(y,t)Q_2\left(\frac{2y}{4-t}-1\right)\,\mathrm{d}y$$
$$\psi_t(\infty)-\psi_t(4)=1. \qquad \text{(A2.4)}$$

In terms of this function the relation (A2.3) can be written as

$$F(s_1,t,u_1)-F(s_2,t,u_2)$$
$$=\frac{(s_1-s_2)(s_1-u_2)(4-t)a_2(t)}{4}$$
$$\times\int_4^\infty \frac{(2x+t-4)\,\mathrm{d}\psi_t(x)}{(x-s_1)(x-u_1)(x-s_2)(x-u_2)Q_2(2x/(t-4)-1)}. \qquad \text{(A2.5)}$$

From this relation bounds on $a_2(t)$ can be derived. For instance, by taking $s_1=3$, $s_2=1$, $t=2$, (A2.5) can be written as

$$F(3,2)-F(1,2)=4a_2(2)\int_4^\infty \frac{\mathrm{d}\psi_2(x)}{(x-3)(x^2-1)Q_2(x-1)}. \qquad \text{(A2.6)}$$

If we define now

$$m=\inf_{x\geqslant 4}\frac{1}{(x-3)(x^2-1)Q_2(x-1)}$$

we obtain from (A2.6) the upper bound

$$a_2(2)\leqslant\frac{F(3,2)-F(1,2)}{4m}.$$

This bound can be evaluated numerically, using the inequalities $F(3,2)<14{\cdot}5$ and $F(1,2)>-7{\cdot}25$ (Lopez and Mennessier 1977) and the value $m=15/2$:

$$0<a_2(2)<0{\cdot}725. \qquad \text{(A2.7)}$$

By applying the same procedure we obtained also the inequalities

$$0<a_2(1)<1{\cdot}5675$$
$$0<a_2(\tfrac{4}{3})<1{\cdot}48$$
$$0<a_2(3)<0{\cdot}2966.$$

If we want to find bounds on the higher partial waves $l\geqslant 4$ we have to write a multiple-subtracted fixed-t dispersion relation for $F(s,t)$, the subtractions being expressed in terms of the first $N/2+1$ partial waves:

$$F(s,t)=\sum_{l=0}^{N}(2l+1)a_l(t)P_l\left(1+\frac{2s}{t-4}\right)+\frac{1}{\pi}\int_4^\infty A(x,t)K_N(x,s,t) \tag{A2.8}$$

the explicit form of the kernel $K_N(x,s,t)$ being known. By now proceeding as in the case of the D wave, we can write (A2.8) in the form

$$F(s,t)=\sum_{l=0}^{N}(2l+1)a_l(t)P_l\left(1+\frac{2s}{t-4}\right)+a_n(t)I_{n,N}(s,t) \tag{A2.9}$$

where the functions

$$I_{n,N}(s,t)=\frac{4-t}{4}\int_4^\infty\frac{K_N(x,s,t)}{Q_n(2x/(4-s)-1)}\mathrm{d}\psi(x)$$

are bounded both from below and above for $0\leqslant s,\ t<4$, and a_n is an arbitrary partial wave with $n\geqslant 2$. From the equation (A2.9) one can easily find bounds for a particular partial wave in terms of $F(s,t)$ and the bounds on the other partial waves at the same point t.

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